Constraints on Quadratic Curves Under Perspective Projection

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ABSTRACT -- In this paper we address the problem of 3D-location estimation based on quadratic-curved features. It is assumed that the true size and true shape of a feature in object space are known, along with its perspective projection in the image plane. In our approach, we decompose the 3D-location-estimation problem into a concatenated process of 3D-orientation estimation and 3D-position estimation. In order to simplify the analysis, we have adopted the concepts of "standard rotation" and "canonical position", introduced by Kanatani. In this context we present a new analytical method for determining the shapes of all quadratic-curved features (whether elliptical, circular, parabolic or hyperbolic), viewed at their "canonical positions". A set of geometrical constraints on the 3D location for each type of quadratic feature is derived, as the main contribution of this research. We prove, in general, that knowledge of the true size and shape does not yield a sufficient number of constraints to determine uniquely 3D orientation and position. As a result, extra constraints must be acquired from other sources and fused with these constraints in order to obtain a unique solution to the 3D-location-estimation problem. The method developed and the results obtained for quadratic-curved features remove ambiguities that exist in a previously developed solution method.

1. Introduction

3D-position-and-orientation estimation of objects in a scene based on various geometrical features is a classical problem both in computer vision and machine vision. This problem has been addressed extensively in the applied literature, where it is referred to as feature-based 3D-location estimation of objects. The body of literature dealing with this general problem is concerned with developing mathematical methods for the estimation of an object's location based on *point* features, whether it is a 3-point problem -- which is also called the triangle pose problem [1], a 4-point problem [2], or an n-point problem [3]. Estimation of the object's location based on *line* features, whose mathematics is similar to that of point features, has also been studied [4].

Recently, Kanatani has addressed the problem based on line and angle features using a new approach [5]. In his approach, it is proposed to transform images of line segments to the center of the image plane (by applying a "standard transformation") as if the camera were rotated -- through a "standard rotation" -- to aim at them. The 3D information extracted at this *"canonical position"* is then transformed back to the original configuration. Kanatani shows that the analysis becomes easier when the image of a line or an angle is moved into a *canonical* position. Based on this approach, and assuming that the true length or the true angle is known, and the projection image is given, he derives the mathematical relations or *constraints* on the 3D-position and orientation of the feature or of the object.

In this paper, we solve the problem for *quadratic-curved features* by adopting Kanatani's concept of "canonical position". That is, we derive the mathematical relations or constraints on the 3D-position and orientation of quadratic-curved features using the "standard rotation" and the "standard transformation" concepts and assuming that the true size and shape of a given quadratic feature is known a priori, with its projection image given. In this context, we introduce an analytical method for estimation of the "standard rotation" and determination of the shape of a quadratic-curved feature at its "canonical position". Furthermore, we show that, *in general*, knowledge of the true shape and size of a quadratic-curved feature does not yield a sufficient number of constraints to determine the 3D-position and orientation uniquely. As a result, *extra* constraints must be acquired from various sources of information and *fused* with the above constraints in order to obtain unique 3D-position and orientation estimates.

Haralick and Chu [6] have developed a general method for 3D-location estimation based on quadratic-curved features. They consider two cases: either the shape and size of a quadratic-curved feature are known, or only its shape is known. For the first case, their method leads to a highly nonlinear optimization problem -- a nonlinear least-squares problem --, with *three* orientation variables; while for the second case, their method leads to a similar nonlinear optimization problem but with *five* variables (three orientation and two size parameters). The main drawback of this method is the residual ambiguity concerning the number of *possible* solutions that

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exist, and the one which is *acceptable*. This is due to the nonlinearity of the optimization process which generally yields a *local* minimum depending on the initial estimates. Nevertheless, the paper still does not provide a geometrical interpretation of the problem and its solutions; furthermore circles, as the most common quadratic feature, are not addressed.

In section 2 of this paper, an analytical method is derived for the estimation of the "standard transformation" required for quadratic curves. In section 3 a set of constraints for different quadratic curves is derived. Furthermore, it is shown that, in general, knowledge of the shape and size of a feature does not yield a sufficient number of constraints to uniquely estimate the 3D-position and orientation of a feature, but, rather, leads to an infinite number of solutions. Section 4 presents a summary and the conclusions of the paper.

2. A Standard Rotational Transformation for Quadratic-Curved Features

Given the perspective projection of a quadratic-curved feature, and the effective focal length of the camera, it is required to estimate a camera-rotation transformation (standard rotation), subsequent to which, the image of the feature would be at the center of the camera's field of view (in the canonical position).

The reason for seeking such a 3D rotation of a camera frame is, as Kanatani points out, to simplify image analysis [5]. Analyzing a feature located at the center of the field of view, i.e., at the center of the image plane, is usually much easier. This has been shown for straight-line and angle features by Kanatani, and in this paper will be shown to be true for quadratic curved features as well.

The fundamental reason for the resulting improvement lies in the *geometrical inhomogeneity* of the image plane under perspective projection. In particular, if a feature is exactly centered in the field of view, the effect of perspective distortion is *minimized*. Conversely, as the offset of the feature with respect to the center of field of view, or the center of the image plane, increases, the perspective distortion is intensified. In some situations, for example in the case of an angle feature or a straight line located at the center of the image plane, the difference between perspective projection and orthographic projection disappears [5].

In the case of quadratic-curved features, addressing the problem at the canonical position simplifies the analysis in the following ways: (1) the constraints, in the form of a set of analytical equations, are of simpler form and of lower order than the set of highly nonlinear equations of higher order (6 and 8 in case of a circular feature) that are obtained if the problem is addressed at other than the canonical position (see [7] for more detail); (2) the intuitive geometrical understanding of the problem and its solution sets is easier; (3) in some situations, as in the case of a circular feature, a 3D problem reduces to a simpler 2D problem; and (4) the derived constraints can be represented easier in a geometrical form; that is, the set of possible solutions can be represented in the form of a 3D envelope.

A basic property of quadratic curves is that their perspective projection is a quadratic curve as well [8-9]. Based on this fact, the first step in our approach is to fit a general quadratic function to the set of boundary points in the image plane and accurately estimate its coefficients [10-11]. Subsequently, based on an optimal fit, we estimate the six parameters of the 3D-location of the feature. This approach allows us to compensate for various distortion factors that are involved in the process of transformation of the curve from object space to computer-image space. From a *practical* point of view, this is very important and leads to a more accurate estimation of 3D-location parameters.

Given the coordinates of the camera's lens center (a vertex) and the equation of the quadratic-curved feature in the image plane (a base), a 3D cone surface can be constructed as follows (see Figure 1). With respect to the image frame, the coordinates of the vertex would be

$$\alpha = 0, \quad \beta = 0, \quad \gamma = -e \,,$$

where, e is the effective focal length of the camera, and the equation of the base is defined as,

$$\begin{cases} F(x',y') \equiv a'x'^2 + 2h'x'y' + b'y'^2 + 2g'x' + 2f'y' + d' = 0 \\ z' = 0 \,. \end{cases} \qquad (1)$$

Then, the 3D cone surface would be [12]:

$$z'^2 F(\alpha,\beta) - z'\gamma \left[x' \frac{\partial F}{\partial \alpha} + y' \frac{\partial F}{\partial \beta} + t \frac{\partial F}{\partial t} \right] + \gamma^2 F(x',y') = 0. \quad (2)$$

In equation (2), t is an auxiliary variable by which $F(x',y')$ is made homogeneous ($F(\alpha,\beta,t)$). The term $t\,(\partial F/\partial t)$ is calculated by first taking the derivative of the homogeneous equation with respect to t and then equating t to unity. Now, to define the equation of the cone with respect to the camera frame, we must use the following transformation:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -e \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}. \qquad (3)$$

Then, the equation of the cone with respect to the camera frame can be expressed as,

$$a\,x^2 + b\,y^2 + c\,z^2 + 2\,f\,y\,z + 2\,g\,z\,x + 2\,h\,x\,y = 0 \,. \qquad (4)$$

In general, the image of the quadratic feature is *not* centered initially. To center the feature in the image plane, that is, to view the feature at its canonical position, we must rotate the z-axis of the camera frame (**n**) such that it would be aligned with the principal axis of the projection cone (**N**). Note that, in general, the principal axis of the projection cone does not pass through the center of the ellipse in the image plane, that is, its base. To perform this alignment, we must use the *Discriminating Cubic* equation [13]: It has been proven that if l_i, m_i, and n_i (i=1,2,3) are the elements of the above-mentioned rotational transformation, the following relations are obtained:

$$\frac{a\,l_i + h\,m_i + g\,n_i}{l_i} = \frac{h\,l_i + b\,m_i + f\,n_i}{m_i} = \frac{g\,l_i + f\,m_i + c\,n_i}{n_i} = \lambda_i \quad (5)$$

where, λ_i are the roots of the following cubic equation:

$$\lambda^3 - \lambda^2 (a+b+c) + \lambda (bc + ca + ab - f^2 - g^2 - h^2) - (abc + 2fgh - af^2 - bg^2 - ch^2) = 0 \ . \quad (6)$$

Equation (6) is called the Discriminating Cubic. Thus, applying the following rotational transformation

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 & 0 \\ m_1 & m_2 & m_3 & 0 \\ n_1 & n_2 & n_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \quad (7)$$

to equation (4), the equation of the cone in its *central* form is obtained as,

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 = 0 \quad (8)$$

where two of the coefficients are always positive, and the third is always negative. If λ_3 is assumed to be the negative coefficient, then, *after* the above *camera rotation*, the camera-feature configuration will be as shown in Figure 2, with the equation of the feature in the image plane being:

$$\frac{X^2}{(e\sqrt{(-\lambda_3/\lambda_1)}\,)^2} + \frac{Y^2}{(e\sqrt{(-\lambda_3/\lambda_2)}\,)^2} = 1 \ . \quad (9)$$

From a geometrical point of view, this rotational transformation transforms a generally *oblique elliptical* cone to a *right elliptical* cone (for a special camera-feature configuration, this leads to a *right circular* cone when $\lambda_1=\lambda_2$). As a result, not only is the analytical solution of the problem generally easier, but, as well, geometrical understanding of the problem and its solution is more profound. This is specially true when a 3D envelope of possible solutions is to be presented.

3. Constraints on Quadratic Curves

In this section, we address the problem of 3D-location estimation when the quadratic feature is viewed at its canonical position. The general approach is to decompose the 3D-location-estimation problem into two separate problems: first the 3D-orientation-estimation problem is addressed; then, based on that, the 3D-position-estimation problem is discussed. Furthermore, it is necessary that there exist geometrical constraints on both the shape and the size of the feature; that is, in each case, it is assumed that the true size and the true shape of a feature in the object space are known, along with its image dimensions and the effective focal length of the camera.

3.1 Circular Features

A circle is a special case of an ellipse, in which the major and minor radii are equal. As a result, a circular feature in a plane is defined in terms of only three parameters: its center coordinates (X_0, Y_0) and its radius (r). Thus, the property of universal symmetry in all directions in a plane reduces the number of orientation parameters to be estimated. But, on the other hand, due to this inherent property of a circular feature, it is *not* possible to determine all three orientation parameters. At best, only its norm can be estimated. As a result, for a circular feature, only five of the six 3D-location parameters can be estimated.

If $\lambda_2 > \lambda_1 > 0$ in equation (8), the major axis of the image of a circular feature, that is an ellipse, would be along the X-axis and its minor axis would be along the Y-axis (See Figure 2).

Subsequent to the standard rotation and the attainment of the canonical position, the problem is much simpler to solve, as can be seen from Figure 2. The camera offset has been compensated *totally* through this rotation. As a result, the problem for a circular feature can be reduced to a problem in 2D-planar geometry. This is shown schematically in Figure 3. The problem can be now defined as estimating the angle α, if R_{min}, R_{max} (the major and minor radii of the ellipse) and e (the effective focal length) are known.

As can be seen from Figure 3, the chord $C'D'$ of the circular feature is mapped onto the major radius of the ellipse in the image plane. Using simple trigonometric relations and principles of Euclidean geometry, we can derive the following relation for the major radius of the ellipse:

$$R_{max} = e \, \frac{r \sin \frac{\delta}{2}}{h - r \sin \alpha \cos \frac{\delta}{2}} \ . \quad (10)$$

As well, we can derive the following two relations for the minor radius of the ellipse:

$$R_{min} = e \, \frac{r(1 - \cos \frac{\delta}{2}) \cos \alpha}{h - r \sin \alpha} \quad (11)$$

$$R_{min} = e \, \frac{r(1 + \cos \frac{\delta}{2}) \cos \alpha}{h + r \sin \alpha} \ . \quad (12)$$

Equating (11) and (12),

$$\cos \frac{\delta}{2} = \frac{r}{h} \sin \alpha \ . \quad (13)$$

Substituting (13) in (11),

$$\frac{h}{r} = \frac{e \cos \alpha}{R_{min}} \ . \quad (14)$$

Using (13) and (14), (10) is simplified as,

$$R_{max} = \frac{e}{\left[\frac{e^2 \cos^2 \alpha}{R_{min}^2} - \sin^2 \alpha \right]^{1/2}} \quad (15)$$

from which an expression for the angle α can be derived:

$$\sin \alpha = \pm \left[\frac{(\frac{e}{R_{min}})^2 - (\frac{e}{R_{max}})^2}{(\frac{e}{R_{min}})^2 + 1} \right]^{1/2} \ . \quad (16)$$

Thus, there exist two symmetrical solutions to the orientation problem (with respect to the Z-axis). Knowledge of the major and minor radii, and the effective focal length is sufficient to estimate them (with π ambiguity). To obtain the unique acceptable solution, we must have an extra geometrical constraint, for example, a constraint that is imposed on the solution by the acquisition of a second image. The estimated angle α determines the orientation of the plane of the circular feature with respect to the *XYZ*-frame. Applying the rotational transformation (7), the orientation of the circular feature plane will be determined with respect to the camera frame.

The above solution has the following properties: it is an *exact* solution, as opposed to the commonly used approximate solution based on the orthographic-projection assumption; the shape constraint is sufficient to estimate the solution, and there is no need to know the size (r) of the feature; and, only the surface normal (i.e., the orientation of the plane) of the feature can be estimated, that is only two out of three degrees of rotational freedom can be determined, as a consequence of the rotation invariance of a circle in a 2D plane.

The position of a circular feature $(0, 0, h)$ with respect to the *XYZ*-frame can be easily estimated, if the size of the feature is known (size constraint), using the following relation:

$$h = \left[\frac{e}{R_{\min}} \cos \alpha \right] r \qquad (17)$$

where r is the radius of the circular feature. To estimate the position with respect to the camera frame, on the other hand, the rotational transformation (7) must be applied.

In case the size of the feature is not known, a second image can be acquired to provide a constraint. The solution of the position-estimation problem based on this constraint has been addressed in [7]. Furthermore, an alternative solution method based on 3D analytical geometry has been derived for circular features in [7] as well.

3.2 Elliptical Features

An ellipse in a plane is defined in terms of five parameters: its center coordinates (X_0, Y_0), its size (A and B, major and minor radii) and its orientation (Θ). Since an ellipse does not have the property of universal symmetry, all three orientation parameters can be defined for this feature.

In this case, as opposed to that for a circle, the 3D-location-estimation problem cannot be reduced to a 2D problem. Thus, we must use 3D analytical geometry. With reference to Figure 2, the problem can be defined as follows: Given the following two surfaces:

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 = 0$$

$$l\,X + m\,Y + n\,Z = p\,, \qquad (18)$$

find the coefficients l, m and n such that the curve generated by the intersection of the two 3D surfaces is an ellipse of radius size A and B. To do so, the following rotational transformation can be used:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} -\frac{m}{\sqrt{l^2+m^2}} & -\frac{l\,n}{\sqrt{l^2+m^2}} & l \\ \frac{l}{\sqrt{l^2+m^2}} & -\frac{m\,n}{\sqrt{l^2+m^2}} & m \\ 0 & \sqrt{l^2+m^2} & n \end{bmatrix} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}. \qquad (19)$$

This transformation is defined such that the new Z-axis (i.e., the Z'-axis) is normal to the plane $l\,X + m\,Y + n\,Z = p$, and the new X-axis (i.e., the X'-axis) is parallel to the intersection line of the image plane and the feature plane (See Figure 4). Applying transformation (19), the equation of the plane would reduce to $Z' = p$; thus the equation of the intersection curve would be of the following form:

$$\left[\frac{\lambda_1 m^2}{l^2+m^2} + \frac{\lambda_2 l^2}{l^2+m^2}\right] X'^2 + \left[\frac{\lambda_1 l^2 n^2}{l^2+m^2} + \frac{\lambda_2 m^2 n^2}{l^2+m^2} + \lambda_3\,(l^2+m^2)\right] Y'^2 + \left[\frac{2\lambda_1 lmn}{l^2+m^2} - \frac{2\lambda_2 lmn}{l^2+m^2}\right] X'Y' + p\left[-\frac{2\lambda_1 lm}{\sqrt{l^2+m^2}} + \frac{2\lambda_2 lm}{\sqrt{l^2+m^2}}\right] X' + p\left[-\frac{2\lambda_1 l^2 n}{\sqrt{l^2+m^2}} - \frac{2\lambda_2 m^2 n}{\sqrt{l^2+m^2}} + 2\,\lambda_3\, n\,\sqrt{l^2+m^2}\right] Y' + p^2\left[\lambda_1 l^2 + \lambda_2 m^2 + \lambda_3 n^2\right] = 0\,. \qquad (20)$$

Equation (20) is the equation of the elliptical feature defined with respect to the $X'Y'Z'$-frame.

On the other hand, we can assume, without loss of generality, that the x_w and y_w axes of the world reference frame (or the object frame) are parallel to the major and minor axes of the elliptical feature and that its origin is at the feature center, in which case the two frames ($X'Y'Z'$ frame and the $x_w\,y_w\,z_w$-frame) transform to each other by just a rotation about the Z'-axis (Θ) and a 3D translation (X'_0, Y'_0, $Z'_0{=}p$). (See Figure 4).

Thus we can define the same elliptical feature as follows:

$$\frac{\left[(X'-X'_0)\cos\Theta + (Y'-Y'_0)\sin\Theta\right]^2}{A^2} + \frac{\left[(X'-X'_0)\sin\Theta + (Y'-Y'_0)\cos\Theta\right]^2}{B^2} = 1$$

or,

$$(A^2\sin^2\Theta + B^2\cos^2\Theta)\,X'^2 + 2\left[(B^2-A^2)\sin\Theta\cos\Theta\right] X'Y' + (A^2\cos^2\Theta + B^2\sin^2\Theta)\,Y'^2 - 2\left[(A^2\sin^2\Theta + B^2\cos^2\Theta)\,X'_0 + (B^2-A^2)\sin\Theta\cos\Theta\,Y'_0\right] X' - 2\left[(A^2\cos^2\Theta + B^2\sin^2\Theta)\,Y'_0 + (B^2-A^2)\sin\Theta\cos\Theta\,X'_0\right] Y' + \left[(A^2\sin^2\Theta + B^2\cos^2\Theta)\,X_0'^2 + (A^2\cos^2\Theta + B^2\sin^2\Theta)\,Y_0'^2 + 2\,(B^2-A^2)\sin\Theta\cos\Theta\,X'_0 Y'_0\right] = A^2B^2. \qquad (21)$$

Normalizing (21) with respect to the coefficient of the X'^2-term, the following two equations can be obtained by equating the coefficients of the $X'Y'$-term and the Y'^2-term in equations (20) and (21):

$$\frac{(\lambda_1 l^2 + \lambda_2 m^2) n^2 + \lambda_3 (l^2 + m^2)^2}{\lambda_1 m^2 + \lambda_2 l^2} = \frac{1 + E^2 \tan^2 \Theta}{E^2 + \tan^2 \Theta} \qquad (22)$$

$$\frac{(\lambda_1 - \lambda_2) lmn}{\lambda_1 m^2 + \lambda_2 l^2} = \frac{(E^2 - 1) \tan \Theta}{E^2 + \tan^2 \Theta} \qquad (23)$$

where, $E = \frac{B}{A}$ (the eccentricity of an ellipse). Furthermore, there exists the following third relation between the unknowns l, m, and n:

$$l^2 + m^2 + n^2 = 1 \ . \qquad (24)$$

Thus, there exist three equations (22 to 24) with four unknowns (l, m, n and Θ). Therefore, it can be concluded that there exist *many* solutions to the orientation-estimation problem, and thus, without an extra geometrical constraint, it would not be possible to find a unique acceptable one. Note that in the special case, when $A = B$ (that is, in the case of a circular feature), because of the rotation-invariance of the circle, the above equations can be simplified to 3 equations in only three unknowns (l, m, and n) and thus are solvable (see [7] for more detail).

For the position-estimation problem for the general case, if it is assumed that by using an extra geometrical constraint, the orientation problem has been first solved, then the values for l, m, n, and Θ are known. Again, by considering equations (20) and (21), the following equations can be derived (note that $Z'_0 = p$)

$$\begin{cases} A_1 Z'_0 &= -2 B_1 X'_0 - 2 C_1 Y'_0 \\ A_2 Z'_0 &= -2 B_2 Y'_0 - 2 C_1 X'_0 \\ A_3 Z'^2_0 &= B_1 X'^2_0 + B_2 Y'^2_0 + 2 C_1 X'_0 Y'_0 - A^2 B^2 \end{cases} \qquad (25)$$

where A_1, A_2, A_3, B_1, B_2, and C_1 are constant terms defined in terms of l, m, n, λ_1, λ_2, λ_3, A, B, and Θ. There will be two solutions of the following form for Z'_0:

$$Z'_0 = \pm \frac{A_4}{B_4} \qquad (26)$$

where A_4 and B_4 have constant values defined in terms of l, m, n, etc. Only the solution in front of the camera is acceptable. That is, we must select Z'_0 such that after the rotation transformation, z_0 would be positive. However, for X'_0 and Y'_0, there exists only one solution, as can be derived from equations (25). To estimate the coordinates with respect to the camera frame, we must apply the rotational transformations (19) and (7).

One last comment that must be made is that, based on what has been derived for elliptical features in general, it is not even possible to obtain a set of possible solutions (which together generate a 3D envelope) without the knowledge of the *size* of the feature (A and B). We emphasize this result, since, unfortunately, it is in marked disagreement with what Haralick and Chu have claimed earlier [6], that is, without using an extra geometrical constraint, or even without knowing the size of a quadratic-curved feature, it is possible to estimate the six parameters of 3D location through optimization. The above conclusion are valid as well for other quadratic-curved features (hyperbolic and parabolic) [14].

4. Summary and Conclusions

The principal objective of this paper has been to determine the set of constraints on the 3D position and 3D orientation of quadratic curves under perspective projection. From what has been shown in the previous sections, we can provide the following brief summary.

In order to simplify the analysis, we have adopted the concepts of "standard rotation" and "canonical position" which were introduced by Kanatani. In this context, an analytical method was developed to estimate the standard rotation, and corresponding to it, to determine the image of a quadratic feature at its canonical position. Furthermore, it has been shown that due to the small angles of view of conventional cameras, hyperbolic and parabolic curves are always mapped onto ellipses in the image plane [14]. To further simplify the problem of 3D-location estimation, it was decomposed into a concatenated process of solution of a 3D-orientation problem and a 3D-position problem.

For circular features, the application of the proposed method reduces the 3D problem to a simple 2D problem. Two unique analytical solutions were obtained for the orientation problem, and, corresponding to each, a unique position solution was derived. For other quadratic curves (the ellipse, hyperbola, and parabola), it was shown that shape and size constraints are not sufficient to provide a unique orientation solution. In general, for these curves, the method leads to three analytical constraints for the 3D-orientation-estimation problem which in turn leads to 3D envelopes of an infinite number of possible solutions. As the result, an *extra* geometrical constraint is required in order to find a unique acceptable one. For position estimation, subsequent to orientation estimation, it was shown that there exists a unique acceptable solution for all quadratic curves.

The result of this paper indicates that, in general, for 3D-location estimation of quadratic-curved features, not only is knowledge of their shape and size required, but as well, an extra geometrical constraint must be used in order to determine a unique solution.

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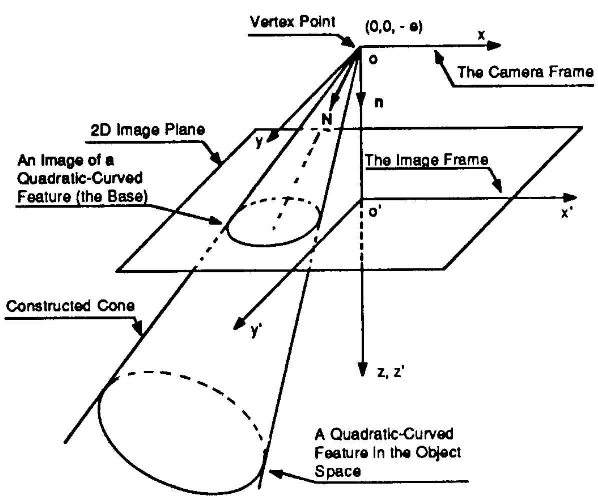

Figure 1. Schematic Representation of the Camera-Feature Configuration.

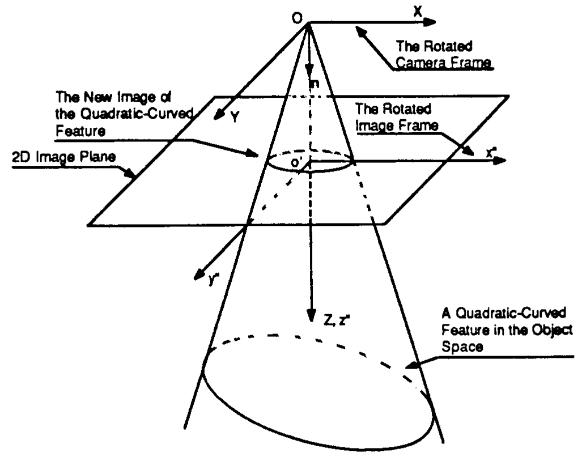

Figure 2. The Camera-Feature Configuration at the Feature's Canonical Position.

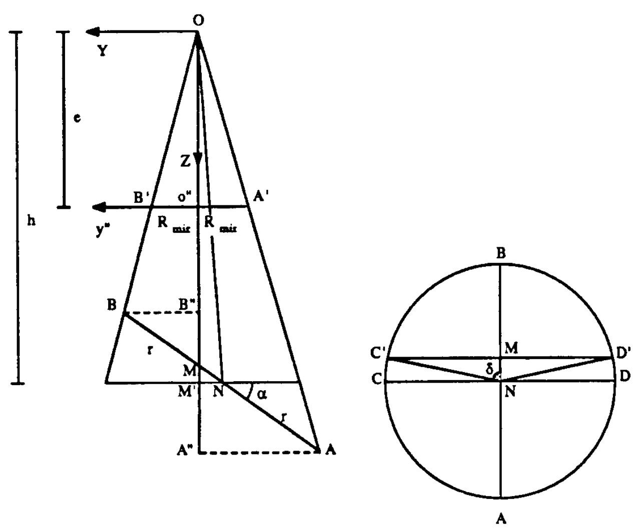

Figure 3. 2D-Schematic Representation of the Circular-Feature Problem.

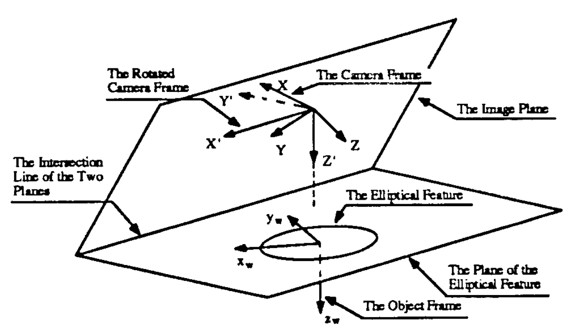

Figure 4. Definitions of the X'Y'Z'-Frame and the World-Reference Frame (the Object Frame).